Goldstone bosons and solitons on the brane*

J. A. R. Cembranos, A. Dobado and A. L. Maroto
Departamento de Física Teórica,
Universidad Complutense de Madrid, 28040 Madrid, Spain

ABSTRACT

In this work we compute the effective action describing the low-energy brane Goldstone-boson dynamics. The Goldstone bosons or branons appear because the brane spontaneously breaks the translational invariance along the extra dimensions. We discuss the Higgs-like mechanism in which the Kaluza-Klein gauge fields absorb the Goldstone bosons and acquire mass. We also present the explicit form of the effective action describing the low-energy interactions between the three-brane Goldstone bosons and the Standard Model fields. In addition to the branons, two new kinds of states arise corresponding to Skyrmion-like solutions and wrapped states. We study the main properties of these states, their geometrical interpretation and the relations between them.

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1 Introduction

The possibility of having large extra dimensions has been considered in recent works as a solution of the hierarchy problem. In this proposal, the most relevant scale would be the Planck scale in $D = 4+n$ dimensions M_D, which is assumed to be not too far from the TeV scale. In the original approach [1], the much larger value of the 4-dimensional Planck constant would be due to the size of the extra dimensions, since we typically have $M_P^2 \simeq R^n M_D^{n+2}$, where R is the radius of the compactified extra dimensions. Phenomenological considerations then require that, whereas gravity can propagate in the D-dimensional bulk space, the ordinary matter fields and gauge bosons be bound to live on a 3-dimensional brane, which would constitute the usual spatial dimensions. The brane tension f is the other important scale in this scenario, f^{-1} being the typical size of the brane fluctuations.

Special attention has been paid to the description of the low-energy sector of the model. This sector would include the Standard Model (SM) fields, the gravitons and the possible excitations of the brane [2]. The presence of extra dimensions allows for the existence, in addition to the standard massless gravitons in four dimensions, of an infinite tower of massive gravitons (Kaluza–Klein [3] gravitons) whose mass is determined by the size of the extra dimensions (see for example [4] for a review of different Kaluza–Klein models). The interaction of the graviton sector with the SM fields has been analysed in a series of papers [5] and different predictions have been obtained that could be tested at future particle colliders. However, in addition to the gravitons, the low-energy spectrum of the theory also contains the brane's own excitations (branons). If the brane has been spontaneously created with a given shape in the bulk (that we will consider as its ground state), the initial isometries of the bulk space could be broken spontaneously by the presence of the brane. The brane configurations obtained by means of some isometry transformations in the bulk will be considered as equivalent ground states and therefore the parameters describing such transformations can be considered as zero-mode excitations of the ground state. When such transformations are made local (depending on the position on the brane), the corresponding parameters play the role of Goldstone bosons (GB) fields of the isometry breaking. Moreover, it has been shown that, in the case where the brane tension f is much smaller than the fundamental scale M_D ($f \ll M_D$), the non-zero KK modes decouple from the GB modes [6, 7] and it is then possible, at least in principle, to make a low-energy effective theory description of the GB dynamics. On the other hand, in the standard Kaluza–Klein models, the isometries in the extra dimensions are understood as gauge transformations in the four-dimensional theory. Therefore, since the GB are associated to the breaking of those *gauge* transformations, it is natural to expect [8] that some kind of Higgs mechanism can take place, which would give mass to the Kaluza–Klein gauge bosons.

In addition to the branons, the brane can support also a new set of topological states. These states are defects that appear due to the non-trivial homotopies of the vacuum manifold [9]. In particular the authors of this reference have considered the case of string and monopole defects on the brane, corresponding to non-trivial first and second homotopy groups.

In this work we are interested in another kind of defects of topological nature related to the Skyrme model [10]. In this model, the baryons are understood as topological solitons that appear in the low-energy pion dynamics described by a chiral lagrangian, the baryon number being identified with the topological charge. This model has provided a very successful description of the baryon properties [11]. The branon effective action mentioned before is formally similar [12] to the chiral lagrangians used for the low-energy description of the chiral dynamics [13] or even the symmetry breaking sector of the Standard Model in the strongly coupled case [14]. Therefore it is quite natural to wonder about the possibility of having chiral solitons (Skyrmions) arising from this effective action. As we will show the answer is positive. In the following we will study in detail those brane-skyrmions, their physical and geometrical interpretation, their main properties and their relation with wrapped states.

The plan of the paper goes as follows: In Sec.2 we introduce our set up and obtain the brane GB effective action starting from a generalized brane action that includes an induced scalar curvature term. This term will be essential in order to determine the brane-skyrmion size. In Sec.3, we describe the Higgs-like mechanism we have commented before. In Sec.4 we derive the coupling of branons with the SM fields. Sec.5 is devoted to the equations for the brane-skyrmions and there we also compute analytically their size and mass in terms of the different parameters involved. In Sec.6 we consider the interacions between the brane-skyrmions and branons and the possible fermionic quantization. In Sec.7 we consider another set of brane states (wrapped states) and study their relation with the brane-skyrmions. Finally, in Sec.8 we set the main results of our work and the conclusions.

2 The effective action for the branons

Let us start by fixing the notation and the main assumptions used in the work. We consider that the four-dimensional space-time M_4 is embedded in a D-dimensional bulk space that for simplicity we will assume to be of the form $M_D = M_4 \times B$, where B is a given N-dimensional compact manifold so that $D = 4 + N$. The brane lies along M_4 and we neglect its contribution to the bulk gravitational field. The coordinates parametrizing the points in M_D will be denoted by (x^μ, y^m), where the different indices run as $\mu = 0, 1, 2, 3$ and $m = 1, 2, ..., N$. The bulk space M_D is endowed with a metric tensor that we will denote by G_{MN}, with signature $(+, -, -...-, -)$. For simplicity, we will consider the following ansatz:

$$G_{MN} \;=\; \left(\begin{array}{cc} \tilde{g}_{\mu\nu}(x) & 0 \\ 0 & -\tilde{g}'_{mn}(y) \end{array} \right). \tag{1}$$

In the absence of the 3-brane, this metric possesses an isometry group that we will assume to be of the form $G(M_D) = G(M_4) \times G(B)$. The presence of the brane spontaneously breaks this symmetry down to some subgroup $G(M_4) \times H$. Therefore, we can introduce the coset space $K = G(M_D)/(G(M_4) \times H) = G(B)/H$, where $H \subset G(B)$ is a suitable subgroup of $G(B)$.

The position of the brane in the bulk can be parametrized as $Y^M = (x^\mu, Y^m(x))$, where we have chosen the bulk coordinates so that the first four are identified with the space-time brane coordinates x^μ. We assume the brane to be created at a certain point in B, i.e. $Y^m(x) = Y^m_0$ which corresponds to its ground state. The induced metric on the brane in such state is given by the four-dimensional components of the bulk space metric, i.e. $g_{\mu\nu} = \tilde{g}_{\mu\nu} = G_{\mu\nu}$. However, when brane excitations (branons) are present, the induced metric is given by

$$g_{\mu\nu} = \partial_\mu Y^M \partial_\nu Y^N G_{MN} = \tilde{g}_{\mu\nu} - \partial_\mu Y^m \partial_\nu Y^n \tilde{g}'_{mn}. \tag{2}$$

For illustrative purposes we show a toy model in Fig. 1 where we have a 1-brane (string) in a $M_3 = M_2 \times S^1$ bulk-space, both in its ground state (flat brane) and in an excited state (wavy brane).

Since the mechanism responsible for the creation of the brane is in principle unknown, we will assume that the brane dynamics can be described by an effective action. Thus, we will consider the most general expression which is invariant under reparametrizations of the brane coordinates. Following the philosophy of the effective lagrangian technique, we will also organize the

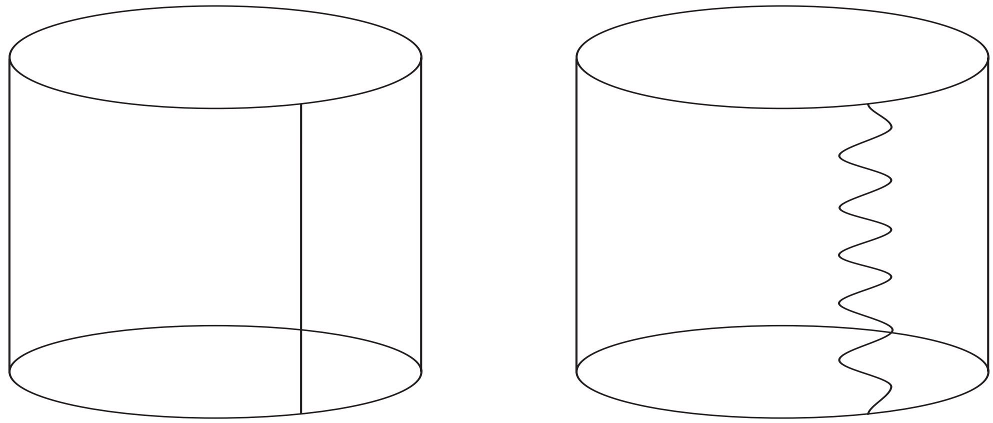

Figure 1: Brane with trivial topology in $M_3 = M_2 \times S^1$. The ground state of the brane is represented on the left. On the right we plot an excited stated.

action as a series in the number of the derivatives of the induced metric over a dimensional constant, which can be identified with the brane tension scale f. Therefore, up to second order in derivatives we find:

$$S_B = \int_{M_4} d^4x\sqrt{g}\left(-f^4 + \lambda f^2 R + \dots\right), \tag{3}$$

where $d^4x\sqrt{g}$ is the volume element of the brane, R the induced curvature and λ an unknown dimensionless parameter. Notice that the lowest order term is the usual Dirac-Nambu-Goto action that was the only one considered in [12].

If the brane ground state is $Y^m(x) = Y_0^m$, the presence of the brane will break spontaneously all the B isometries, except those that leave the point Y_0 unchanged. In other words the group $G(B)$ is spontaneously broken down to $H(Y_0)$, where $H(Y_0)$ denotes the isotropy group (or little group) of the point Y_0. Let H_i be the H generators ($i = 1, 2, ... h$), X_α ($\alpha = 1, 2, ... k = \dim G(B) - \dim H$) the broken generators, and $T = (H, X)$ the complete set of generators of $G(B)$. A similar separation can be done with the Killing fields. We will denote ξ_i those associated to the H_i generators, ξ_α those corresponding to X_α and by $\xi_a(y)$ the complete set of Killing vectors on B. The excitations of the brane along the (broken) Killing fields directions of B correspond to the zero modes and they are parametrized by the branon fields $\pi^\alpha(x)$ that can be understood as coordinates on the coset manifold $K = G(B)/H$. Thus, for a position-independent ground state Y_0^m, the action of an element of $G(B)$ on B will map Y_0 into some other point with coordinates:

$$Y^m(x) = Y^m(Y_0, \pi^\alpha(x)) = Y_0^m + \frac{1}{kf^2}\xi_\alpha^m(Y_0)\pi^\alpha(x) + \mathcal{O}(\pi^2), \qquad (4)$$

where we have set the appropriate normalization of the branon fields and Killing fields with $k^2 = 16\pi/M_P^2$. It is important to note that the coordinates of the transformed point depend only on $\pi^\alpha(x)$, i.e. on the parameters of the transformations corresponding to the broken generators. The rest of the transformations (corresponding to the H subgroup) leave the vacuum unchanged and therefore they are not GB. Thus not all the isometries will give rise to zero modes of the brane. When the B space is homogeneous, the isotropy group does not depend on the particular point we choose, i.e, $H(Y_0) = H$ and it is possible to prove that B is diffeomorphic to $K = G(B)/H$, i.e. the number of GB equals the dimension of B.

According to the previous discussion, we can write the brane effective action in terms of branon fields using:

$$\partial_\mu Y^m(x) = \frac{\partial Y^m}{\partial \pi^\alpha}\partial_\mu \pi^\alpha = \frac{1}{kf^2}\xi_\alpha^m(Y_0)\partial_\mu\pi^\alpha + \mathcal{O}(\pi^2) \qquad (5)$$

and, therefore, introducing the tensor $h_{\alpha\beta}(\pi)$ as

$$h_{\alpha\beta}(\pi) = f^4 \tilde{g}'_{mn}(Y(\pi))\frac{\partial Y^m}{\partial \pi^\alpha}\frac{\partial Y^n}{\partial \pi^\beta}, \qquad (6)$$

we have

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} - \frac{1}{f^4}h_{\alpha\beta}(\pi)\partial_\mu\pi^\alpha\partial_\nu\pi^\beta \qquad (7)$$

Thus, for small brane excitations in a background metric $\tilde{g}_{\mu\nu}$, the effective action becomes

$$S_{eff}[\pi] = S_{eff}^{(0)}[\pi] + S_{eff}^{(2)}[\pi] + S_{eff}^{(4)}[\pi] + ... \qquad (8)$$

where:

$$S_{eff}^{(0)}[\pi] \;\; = \;\; -f^4\int_{M_4} d^4x\sqrt{\tilde{g}}. \qquad (9)$$

The $\mathcal{O}(p^2)$ contribution is the non-linear sigma model corresponding to a symmetry-breaking pattern $G \rightarrow H$ plus the background scalar curvature term:

$$S_{eff}^{(2)}[\pi] \;\; = \;\; \frac{1}{2}\int_{M_4} d^4x\sqrt{\tilde{g}}h_{\alpha\beta}\partial_\mu\pi^\alpha\partial^\mu\pi^\beta + \lambda f^2\int_{M_4} d^4x\sqrt{\tilde{g}}\tilde{R}. \qquad (10)$$

We are assuming that the branons derivative terms are of the same order as those with metric derivatives. The fourth-order term is obtained by expanding both the metric determinant and the induced scalar curvature in branon fields. Up to a surface term (see [15]), we obtain:

$$
\begin{aligned}
S^{(4)}_{eff}[\pi] &= \frac{-1}{8f^4}\int_{M_4} d^4x\sqrt{\tilde{g}}h_{\alpha\beta}h_{\gamma\delta}(\partial_\mu\pi^\alpha\partial^\mu\pi^\beta\partial_\nu\pi^\gamma\partial^\nu\pi^\delta - 2\partial_\mu\pi^\alpha\partial^\nu\pi^\beta\partial_\nu\pi^\gamma\partial^\mu\pi^\delta) \\
&+ \frac{\lambda}{2f^2}\int_{M_4} d^4x\sqrt{\tilde{g}}h_{\alpha\beta}\partial^\mu\pi^\alpha\partial^\nu\pi^\beta(2\tilde{R}_{\mu\nu} - \tilde{R}\tilde{g}_{\mu\nu}), \qquad (11)
\end{aligned}
$$

Let us emphasize again that the above effective action is an expansion in branon fields (or metric) derivatives over f^2 and not an expansion in powers of π fields, i.e, it is a low-energy effective action. Notice also that we have assumed that the $G(M_D)$ symmetry is exact, which implies that the branon fields are massless. However, in a real situation, such symmetry will be only approximately realized. In this case, we will expect the branons to acquire a mass that will measure the breaking of the $G(M_D)$ symmetry. In [15] we have studied these effects and how the symmetry breaking affects the brane ground state.

3 Kaluza–Klein gauge bosons and the Higgs mechanism

As commented in the introduction, the isometries in the B space are considered as gauge transformation in the Kaluza–Klein theories [4]. In this section we study under which circumstances the GB associated to the isometry breaking can give rise to the longitudinal components of the Kaluza–Klein gauge bosons, as in the standard Higgs mechanism.

We start with the Hilbert–Einstein action for the gravitational field in D dimensions plus the brane action: $S = S_G + S_B$, i.e.

$$
S = \frac{-1}{16\pi G_D}\int_{M_D} d^Dz\sqrt{G}R_D - f^4\int_{M_4} d^4x\sqrt{g}G_{MN}g^{\mu\nu}\partial_\mu Y^M\partial_\nu Y^N, \qquad (12)
$$

where $z = (x, y)$ are the coordinates defined on M_D, x and y being the coordinates defined on M_4 and B respectively, and R_D is the D-dimensional scalar curvature.

In a general non-Abelian case, we consider the usual ansatz for the metric tensor used in the Kaluza–Klein theories:

$$
G_{MN} = \begin{pmatrix} \tilde{g}_{\mu\nu}(x) - g'_{mn}(y)B^m_\mu(x,y)B^n_\nu(x,y) & B^n_\mu(x,y) \\ B^m_\nu(x,y) & -g'_{mn}(y) \end{pmatrix},
$$

where $B^n_\mu(x,y) = \xi^n_a(y)A^a_\mu(x)$ with $\xi^n_a(y)$ the Killing vectors corresponding to the isometry group $G(B)$ introduced above. Now the gauge transformations are $y^m \to y'^m = y^m + \xi^m_a(y)\epsilon^a(x)$. As is well known in this case, the gravitational action S_G can be written as

$$S_G = \frac{-1}{16\pi G_N}\int d^4x\sqrt{\tilde{g}}\tilde{R} - \frac{\langle \xi^n_a \xi^m_b g'_{mn}\rangle}{16\pi G_N}\frac{1}{4}\int_{M_4} d^4x\sqrt{\tilde{g}}F^a_{\mu\nu}F^{\mu\nu b}, \tag{13}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - C_{abc}A^b_\mu A^c_\nu$ and

$$16\pi G_N = \frac{16\pi G_D}{\int_B d^{D-4}y\sqrt{g'}}. \tag{14}$$

In order to obtain the standard Yang–Mills action, the normalization of the Killing vectors is given by

$$\langle \xi^m_a(y)\xi^n_b(y)g'_{mn}(y)\rangle = k^2\delta_{ab}, \tag{15}$$

where $g'_{mn}(y)$ is the B space-time metric, again $k^2 = 16\pi G_N$ and the brackets are defined as the B manifold average [13]

$$\langle f(y)\rangle = \frac{\int_B d^{D-4}y\sqrt{g'}f(y)}{\int_B d^{D-4}y\sqrt{g'}}. \tag{16}$$

The induced metric in this case with non-vanishing gauge fields is given by:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} - \Delta_\mu Y^m \Delta_\nu Y^n g'_{mn} \tag{17}$$

where the covariant derivative is defined as

$$\Delta_\mu Y^m = \partial_\mu Y^m - \xi^m_a A^a_\mu, \tag{18}$$

which can be written as

$$\Delta_\mu Y^m = \frac{\partial Y^m}{\partial \pi^\alpha}\partial_\mu \pi^\alpha - \xi^m_a A^a_\mu = \frac{1}{kf^2}\xi^m_\alpha(Y_0)(\partial_\mu\pi^\alpha - kf^2 A^\alpha_\mu) - \xi^m_i(Y_0)A^i_\mu + \mathcal{O}(\pi^2). \tag{19}$$

Since the i indices correspond to the generators of the isotropy group $H(Y_0)$, the Killing fields vanish at Y_0, i.e. $\xi^m_i(Y_0) = 0$ and the last term vanishes.

Therefore the brane action S_B is

$$S_B = -f^4\int_{M_4} d^4x\sqrt{\tilde{g}} + \frac{1}{2}\int_{M_4} d^4x\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}h_{\alpha\beta}D_\mu\pi^\alpha D_\nu\pi^\beta + \mathcal{O}(\pi^4). \tag{20}$$

where $D_\mu \pi^\alpha = \partial_\mu \pi^\alpha - k f^2 A_\mu^\alpha$. Thus the gauge boson mass matrix is

$$M^2_{\alpha\beta} = k^2 f^4 h_{\alpha\beta}(0). \tag{21}$$

Remember that $Y^m(x) = Y_0^m$ corresponds to $\pi^a = 0$. As commented on before, not all the the gauge bosons will acquire a mass by this mechanism. Only those associated to the broken X_α generators will, their number being determined by the dimension of the $K = G/H$ space. In any case, two important comments are in order. First, the gauge boson masses are quite small whenever $f \ll M_P$. On the other hand it should be remembered that in the standard KK picture, having gauge couplings g small enough to have a sensible weak coupling limit (say $g < 1$) requires having extra dimensions of a typical size of the order of the Planck length, since g^2 is of the order of k^2/R^2 (see for instance [4]). Thus, for the interesting case of large extra dimensions and $f \ll M_D$, graviphotons can be considered massless and decoupled from the rest of the low-energy particles. In this case we can safely assume that the Higgs mechanism has not taken place and the GB can be considered as the only relevant low-energy new degrees of freedom.

4 Couplings to the Standard Model fields

As we have shown in the previous sections, the induced metric on the brane depends on both the four-dimensional bulk metric components $\tilde{g}_{\mu\nu}$ and the Goldstone bosons π^α. In the following, we will assume that the physical space-time metric is $\tilde{g}_{\mu\nu}$, whereas the contribution of the GB to the brane metric can be detected only through their couplings to the Standard Model fields. In order to obtain such couplings, we use the Sundrum effective action for the SM fields [2], which is basically the SM action defined on a curved space-time M_4 whose metric is the induced metric $g_{\mu\nu}$. Let us give the results up to $\mathcal{O}(p^2)$ for the different kinds of fields which have been obtained in [12]:

Scalars

For a scalar field we have:

$$\begin{aligned} S_\Phi &= \frac{1}{2}\int_{M_4} d^4x\sqrt{g}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi = \frac{1}{2}\int_{M_4} d^4x\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \\ &+ \frac{1}{2f^4}\int_{M_4} d^4x\sqrt{\tilde{g}}h_{\alpha\beta}(\pi)\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\pi^\alpha\partial^\nu\pi^\beta \\ &- \frac{1}{4f^4}\int_{M_4} d^4x\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\tilde{g}^{\rho\sigma}h_{\alpha\beta}(\pi)\partial_\rho\pi^\alpha\partial_\sigma\pi^\beta + \mathcal{O}(p^4). \end{aligned} \tag{22}$$

Fermions

The introduction of fermions on the brane is more complicated and we refer the reader to the above reference for details. The results are:

$$\begin{aligned} S &= \int d^4x\sqrt{\tilde{g}}i\bar{\psi}\tilde{\not{D}}\psi - \frac{i}{2f^4}\int d^4x\sqrt{\tilde{g}}h_{\alpha\beta}(\pi)\tilde{g}^{\mu\nu}\partial_\mu\pi^\alpha\partial_\nu\pi^\beta\bar{\psi}\tilde{\not{D}}\psi \\ &+ \frac{i}{2f^4}\int d^4x\sqrt{\tilde{g}}\bar{\psi}h_{\alpha\beta}(\pi)\tilde{g}^{\mu\nu}\partial_\mu\pi^\alpha\ \not{\partial}\pi^\beta\tilde{D}_\nu\psi \\ &- \frac{i}{4f^4}\int d^4x\sqrt{\tilde{g}}\bar{\psi}h_{\alpha\beta}(\pi)\tilde{g}^{\mu\nu}(\not{\partial}(\partial_\mu\pi^\alpha\partial_\nu\pi^\beta) - \partial_\mu(\not{\partial}\pi^\alpha\partial_\nu\pi^\beta))\psi. \end{aligned} \tag{23}$$

where $\tilde{D}_\mu$ is the fermionic covariant derivative corresponding to the background metric $\tilde{g}_{\mu\nu}$. In particular, this way of introducing the couplings of Goldstone bosons to fermions allows us to consider chiral fermions in a straightforward way.

Gauge bosons

For the Yang–Mills action on the brane we can follow similar steps, and we get:

$$\begin{aligned} S_{YM} &= \frac{\mathrm{tr}}{2g^2}\int d^4x\sqrt{\tilde{g}}\tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma}G_{\mu\nu}G_{\rho\sigma} \\ &- \frac{\mathrm{tr}}{4g^2f^4}\int d^4x\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}h_{\alpha\beta}(\pi)(\partial_\mu\pi^\alpha\partial_\nu\pi^\beta)\tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma}G_{\mu\nu}G_{\rho\sigma} \\ &+ \frac{\mathrm{tr}}{g^2f^4}\int d^4x\sqrt{\tilde{g}}h_{\alpha\beta}(\pi)(\partial_\lambda\pi^\alpha\partial_\kappa\pi^\beta)\tilde{g}^{\mu\lambda}\tilde{g}^{\rho\kappa}\tilde{g}^{\nu\sigma}G_{\mu\nu}G_{\rho\sigma} \end{aligned} \tag{24}$$

From the above discussion we see that the GB always interact by pairs with the SM matter. In addition, due to their geometric origin, those interactions are very similar to the gravitational interactions since the π fields couple to all the matter fields with the same strength, which is suppressed by a factor f^4. This is quite interesting since it could explain why they have not been observed so far, provided they exist at all. However, moderate values of the brane tension around the TeV scale could make their production possible in the next generation of colliders.

5 Brane-skyrmions

The branon fields introduced in the previous sections describe small oscillations of the brane around its ground state. Thus there is some similitude with the well known chiral lagrangian approach in which a non-linear sigma model (NLSM) is used to describe the low-energy pion dynamics. Apart from pions, the NLSM can also be used to describe other non-trivial states in the hadron spectrum such as baryons. For that purpose, the non-trivial topological structure of the coset space K plays a fundamental role. In fact, baryons can be identified with certain topologically non-trivial maps between the (compactified) space S^3 and the coset manifold K known as Skyrmions.

Let us then consider static branon field configurations with finite energy, that accordingly vanish at the spatial infinity. Thus, we can compactify the spatial dimensions to S^3 and the static configurations will be mappings: $\pi^\alpha : S^3 \rightarrow K$. Therefore, these mappings can be classified according to the third homotopy group of K, i.e. $\pi_3(K)$. As a consequence, mappings belonging to different non-trivial homotopy classes cannot be deformed in a continuous fashion from one to the other. This implies that such configurations cannot evolve in time classically into the trivial vacuum $\pi = 0$ and therefore they are stable states. For the sake of simplicity we will consider the case in which we have $N = 3$ extra dimensions with $B = S^3$. In this case, since B is an homogeneous space, we have $K \sim B = S^3 \sim SU(2)$, i.e the coset manifold is also a 3-sphere. Thus, we will have $\pi_3(S^3) = \mathbf{Z}$ and the mappings can be classified by an integer number, usually referred to as the winding number n_W. We will also assume in the following, that the background metric is flat, i.e, $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$.

For static configurations the brane-skyrmion mass can be obtained directly from the effective lagrangian as:

$$M[\pi] = -\int d^3x \mathcal{L}_{eff}. \tag{25}$$

In general this expression will be divergent because of the volume contribution coming from the lowest order term $S^{(0)}_{eff}[\pi]$ which reflects the fact that the brane has infinite extension with finite tension. Therefore, in order to obtain a finite skyrmion mass, we will substract the vacuum energy $M[0]$, i.e.:

$$M_S[\pi] = M[\pi] - M[0] = f^4 \int_{M_3} d^3x\sqrt{g} - \lambda f^2 \int_{M_3} d^3x \sqrt{g} R - M[0]. \tag{26}$$

In other words we are defining the mass of the brane-skyrmion as the mass of the brane with the topological defect minus the mass of the brane in its ground state with no topological defect.

In order to simplify the calculations we will introduce spherical coordinates on both spaces, M_4 and K. In M_4 we denote the coordinates $\{t, r, \theta, \varphi\}$

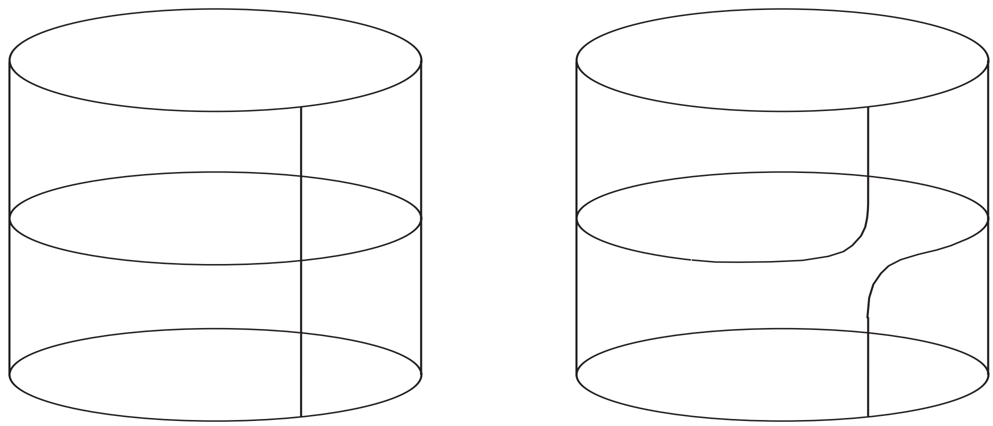

Figure 2: Brane configurations with $n_W = 1$ in $M_3 = M_2 \times S^1$. On the right we plot a non-zero size skyrmion. On the left a zero-size 1-brane-skyrmion is shown. It has the same mass and shape as a wrapped soliton and a topologically trivial (world) brane. However the topology is not the same.

with $\phi \in [0, 2\pi)$, $\theta \in [0, \pi]$ and $r \in [0, \infty)$. On the coset manifold K, the spherical coordinates are denoted $\{\chi_K, \theta_K, \phi_K\}$ with $\phi_K \in [0, 2\pi)$, $\theta_K \in [0, \pi]$ and $\chi_K \in [0, \pi]$. These coordinate are related to the physical branon fields (local normal geodesic coordinates on K) by:

$$\begin{aligned}
\pi_1 &= v \sin \chi_K \sin \theta_K \cos \phi_K, \\
\pi_2 &= v \sin \chi_K \sin \theta_K \sin \phi_K, \\
\pi_3 &= v \sin \chi_K \cos \theta_K.
\end{aligned} \tag{27}$$

The coset metric $h_{\alpha\beta}$ is the usual S^3 metric in spherical coordinates. In these coordinates, the brane-skyrmion with winding number n_W is given by the non-trivial mapping $\pi^\alpha : S^3 \longrightarrow S^3$ defined from:

$$\begin{aligned}
\phi_K &= \phi, \\
\theta_K &= \theta, \\
\chi_K &= F(r),
\end{aligned} \tag{28}$$

with boundary conditions $F(0) = n_W \pi$ and $F(\infty) = 0$. This map is usually referred to as the hedgehog ansatz (see Fig. 2).

In order to calculate the brane-skyrmion mass from (26), we need explicit expressions for the induced metric determinant and the scalar curvature in terms of the profile function $F(r)$, so that we can write M_S as a functional of $F(r)$. The actual mass of the brane-skyrmion with winding number n_W

will be obtained by minimizing the functional $M_S[F]$ (26) in the space of functions $F(r)$ with the appropriate boundary conditions. This problem is in general rather complicated, but we can obtain an upper bound to the mass by using a family of functions parametrized by a single parameter, and minimizing with respect to that parameter. In particular, it is very useful to work with the Atiyah-Manton ansatz [16]:

$$F(r) = n_W \pi \left(1 - \frac{1}{\sqrt{1 + L^2/r^2}} \right). \tag{29}$$

By minimizing the brane-skyrmion mass with respect to L we will get: $M_S \equiv min_L\ M_S(L) \equiv M_S(L_m)$ for different values of the parameter λ. Thus:

- For $\lambda = 0$ and using the ansatz above for $n_W = 1$ we find that $M_S(L)$ is minimized for $L_m = 0$. The corresponding mass is given by

$$M_S = 2\pi^2 f^4 R_B^3, \tag{30}$$

  i.e. the brane-skyrmion describes a pointlike particle with a finite mass given by the volume of the extra dimensions times the brane tension f^4. It can be shown that this result is general, i.e. it is valid for any parametrization of the $F(r)$ function and not only for the Atiyah-Manton one.

- For $\lambda > 0$, we find that $M_S(L)$ is minimized for some $L_m > 0$ (non-zero size brane-skyrmion). In this case, assuming that the contribution from the curvature term never becomes negative, it is possible to obtain the following lower bound on the mass $M_S > M_S(\lambda = 0) = 2\pi^2 f^4 R_B^3$. An upper bound can be obtained evaluating M_S in the limit $L = 0$, simply assuming that this limit is well defined so that we can commute the limit with the integration. Thus

$$M_S < M_S(L = 0) = 2\pi^2 f^4 R_B^3 \left(1 + 6 \frac{\lambda}{R_B^2 f^2} \right). \tag{31}$$

  These results are general for any monotonic parametrization and, in particular, we have checked numerically that they hold for the Atiyah-Manton case.

- Finally, for $\lambda < 0$, we find again that the minimum M_S corresponds to zero size brane-Skyrmion and the corresponding mass is

$$M_S = 2\pi^2 f^4 R_B^3 \left(1 + 6 \frac{\lambda}{R_B^2 f^2} \right). \tag{32}$$

When $\lambda < -R_B^2 f^2/6$ the brane-skyrmion mass becomes negative. In this case, using non-monotonic parametrizations, we have obtained that the mass is actually not bounded from below, since the curvature term can be made arbitrarily large. As a consequence only in this last case, the brane-skyrmion becomes unstable. These results are summarized in Table.1

λ	Size	Mass
$\lambda = 0$	$L_m = 0$	$M_S = 2\pi^2 f^4 R_B^3$
$\lambda > 0$	$L_m > 0$	$2\pi^2 f^4 R_B^3 < M_S < 2\pi^2 f^4 R_B^3(1 + 6\frac{\lambda}{R_B^2 f^2})$
$-R_B^2 f^2/6 < \lambda < 0$	$L_m = 0$	$M_S = 2\pi^2 f^4 R_B^3(1 + 6\frac{\lambda}{R_B^2 f^2})$
$\lambda < -R_B^2 f^2/6$	$L_m = 0$	$M_S \to -\infty$

Table 1: Values of the size L_m and mass M_S for the brane-skyrmion with $n_W = 1$ for different values of the λ parameter.

In Fig.3 we show the brane-skyrmion mass as a function of λ and L. We see that the minimum is displaced from $L = 0$ when $\lambda > 0$.

For more precise numerical results see [15]. An interesting conclusion that can be obtained from those results is that the interaction between two classical brane-skyrmions with topological number $n_W = 1$ is repulsive when their sizes are non zero, whereas they do not interact if their sizes are exactly zero.

We can also take into account the effects of the possible branon mass on the Skyrmion properties. As shown in [15], the presence of the branon masses does not affect the brane-skyrmion stability, the only difference with the massless case being that the brane-skyrmion mass increases.

Finally this analysis can be extended to higher-dimensional Skyrmions. For the simple case where $M_D = M_{N+1} \times S^N$ similar results hold, except for $N = 1$ where brane-Skyrmions are unstable, and $N = 2$ where the contribution of the curvature term vanishes.

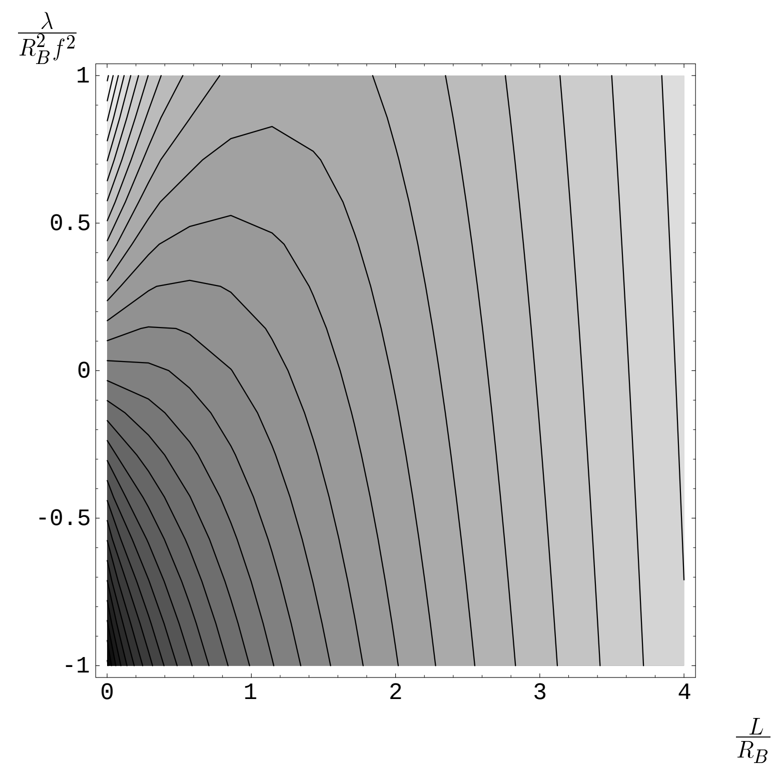

Figure 3: Contour plot of the brane-skyrmion mass as a function of $\lambda/(R_B^2 f^2)$ and L/R_B.

6 Interaction Lagrangian and fermionic quantization

In this section we will study the interaction between the brane-skyrmions and the branons. For simplicity we will consider only the case $M_7 = M_4 \times S^3$ with $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ so that $K \simeq SU(2)$. Then we can follow the well known steps for quantization of the standard chiral dynamics Skyrmion [17]. It is possible to split the isometry group G as $G = SU(2)_L \times SU(2)_R$ and H corresponds to the isospin group $SU(2)_{L+R}$. The parametrization of the coset is usually done in terms of a $SU(2)$ matrix $U(x)$ and the Skyrmion is usually written as

$$\begin{aligned} U(x) &= \exp(iF(r)\hat{x}^a\tau^a) = \cos F(r) + i\tau^a\hat{x}^a \sin F(r) \\ &= \pm\sqrt{1-\frac{\pi^2}{v^2}} + i\tau^a\hat{x}^a \sin F(r), \end{aligned} \tag{33}$$

where τ^a are the $SU(2)$ generators. From this expression we can identify the Goldstone bosons fields $\pi^\alpha = v \sin F(r)\hat{x}^\alpha$. The quantization of the isorotations of the Skyrmion solution (which correspond to rotations in the compactified space $B = S^3$ in our case) requires the well-known relation $J = I$ where J and I are the spin and the isospin indices. In principle the allowed values of J are $J = 0, 1/2, 1, 3/2, ...$ As explained by Witten [11], fermionic quantization is possible because of the Wess-Zumino-Witten (WZW) term $k\Gamma$ with k integer, that can be added to the Goldstone boson effective action. For k even the Skyrmion is a boson, but for k odd it is a fermion. For the $SU(2)$ case, the functional Γ has no dynamics and becomes a topological invariant related with the homotopy group $\pi_4(SU(2)) = \mathbf{Z_2}$. Note that for a suitable compactification of the space-time this is the relevant group for the Goldstone boson map. A map belonging to the non-trivial class could for example describe the creation of a Skyrmion-antiskyrmion pair followed by a 2π rotation of the Skyrmion and finally a Skyrmion-antiskyrmion annihilation. In the fermionic case this field configuration must be weighted with a -1 in the Feynman path integral. For an adiabatic 2π rotation of the Skyrmion around some axis, the WZW term contributes $k\pi$ to the action and $(-)^k$ to the amplitude which can be understood as an $exp(i2\pi J)$ factor. Therefore both possibilities, bosonic and fermionic quantization of the Skyrmion, are open. In principle this result can also be extended to the more general case of S^N brane-skyrmions considered in the last section where $M_D = M_{N+1} \times S^N$ with $D = 2N + 1$ since $\pi_{N+1}(S^N) = \mathbf{Z_2}$ for $N \geq 3$.

In order to study the low-energy interactions of the brane-skyrmions with the branons we have to obtain the appropriate effective lagrangian. This lagrangian must be $G(B)$ symmetric and the brane-skyrmion is described in it by a complex field because of the topological charge. Thus for example this field will be a complex scalar Φ for $J = 0$ or a Dirac spinor Ψ for $J = 1/2$. For the scalar case the invariant lagrangian with the lowest number of derivatives can be written as:

$$\mathcal{L}_s = \alpha \Phi^* \Phi h_{\alpha\beta}(\pi) \partial_\mu \pi^\alpha \partial^\mu \pi^\beta. \tag{34}$$

The coupling α can be obtained from the large distance behaviour of the branon field in the brane-skyrmion configuration, i.e., $F(r) \simeq B/r^2$. In particular for the Atiyah-Manton ansatz with $n_W = 1$, we get $B = L^2\pi/2$. By using the lagrangian $\mathcal{L}_s$ it is also possible to obtain the branon field produced by the brane-skyrmion field Φ and by comparison with the above results we arrive at [18]

$$\alpha = -\frac{8}{3}\pi^2 v^2 B^2 = -\frac{2}{3} v^2 \pi^4 L_m^4. \tag{35}$$

From this lagrangian it is possible, for example, to compute the cross sections for producing a brane-skyrmion-antibrane-skyrmion pair from two branons.

The fermionic case can be studied in a similar way [18, 17] although a consistent analysis would be more involved, since it requires the quantization of the rotational modes.

7 Wrapped states

In this section we introduce another kind of states which can appear as topological excitations of the branes. These states correspond to brane configurations wrapped around the compactified spaces B which typically will be assumed to be S^N for $M_D = M_{N+1} \times B$. A given wrapped states is located at some well defined point of the space M_N. The possibility of having this wrapped states is related to the homotopy group $\pi_N(B) = \mathbf{Z}$ which is obviously the case for $B = S^N$ but also for other spaces. In principle, wrapped states can be present even when there is no world brane, i.e., when we do not have a brane extended along the big space M_N. However, one of the most interesting cases occurs when the wrapped states are located at one point of the world brane and then can be understood as world-brane excitations. Note that as far as the relevant homotopy group is again $\mathbf{Z}$ we have also antiwrapped states which correspond to negative winding numbers. Thus a world brane can get excited by creating a wrapped-antiwrapped state at some given point. In Fig. 4 we show a single wrapped state at rest (left) and branon-excited. On the left of Fig.2 we show a wrapped state (circle) located at one point of the world brane (straight line) for the case $N = 1$.

To study in more detail the main properties of these wrapped states we concentrate now on a four-dimensional space-time M_4 embedded in a 7-dimensional bulk space that we are assuming to be $M_7 = M_4 \times B$ with $M_4 = \mathbf{R} \times M_3$ and $B = S^3$. Now, unlike the brane-skyrmion case, the finite energy requirement do not lead to any compactification of the world space M_3 because the brane is going to be wrapped around B which is compact. However, for technical reasons it is still useful to compactify M_3 to S^3 by adding the spatial infinite point. The wrapped brane produces the spontaneous breaking of the M_7 isometry group, which we assume to be $G(M_7) = G(\mathbf{R} \times M_3 \times S^3) = G(\mathbf{R} \times S^3) \times G(M_3)$ to the $G(\mathbf{R} \times S^3) \times H'$

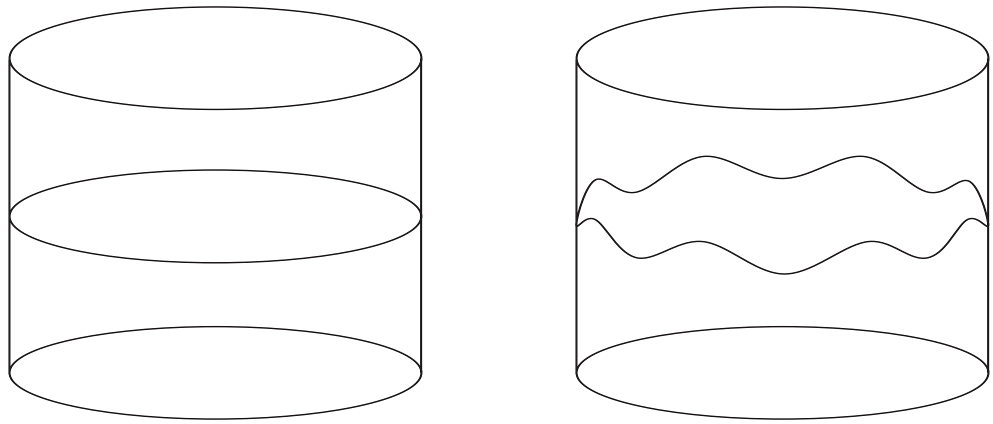

Figure 4: Wrapped brane with topology number 1 in a 1-dimensional compact extra-space in addition to 1-dimensional space. The ground state of the brane is represented in the left graphic, and a exited state is represented in the right graphic.

group where H' is the isotropy group of M_3 which is assumed to be homogeneous. Then the coset space is defined by $K' = G(M_3)/H'$. In the simplest case $M_3 = S^3$ we have $K' = SO(4)/SO(3) = S^3$ and then $K' \sim B \sim S^3$. Therefore the low-energy brane excitations can be parameterized as

$$\begin{aligned} \Pi : B &\longrightarrow K' \\ y &\longrightarrow \pi(y), \end{aligned} \tag{36}$$

where y are coordinates on B and π are coordinates on K'. On the other hand, as far as the quotient space K' and M_3 are both topologically equivalent to S^3, it is possible to describe the wrapped brane by giving its position on the M_3 space X^i as a function of y^m, i.e. $X^i = X^i(y^m)$. In particular it is possible to choose the coordinates so that

$$X^i(y) = \frac{1}{f^2}\delta^i_\alpha \pi^\alpha(y) + ... \tag{37}$$

locally. In the following we will use X^i instead of π^α to label the wrapped-brane points in terms of the brane parameters y . Let us now rearrange the coordinates for convenience in order to have $Y^M = (t, y^m, X^i(y))$ where t is the temporal coordinate $t = x^0$. The bulk metric is then:

$$G_{MN} = \left(\begin{array}{ccc} \tilde{g}_{00} & & 0 \\ & -\tilde{g}'_{mn}(y) & \\ 0 & & -\tilde{g}_{ij}(x) \end{array} \right) = \left(\begin{array}{cc} \tilde{g}'_{rs}(y) & 0 \\ 0 & -\tilde{g}_{ij}(x) \end{array} \right), \tag{38}$$

where $\tilde{g}'_{rs}$ is the background metric on the space-time manifold $\mathbf{R} \times B$, i.e., $r, s = 0, 1, 2, 3$. The induced metric on this manifold can be evaluated in a way similar to the brane-skyrmion case. Thus in the ground state, the induced metric on the wrapped brane is given by the four-dimensional components of the bulk space metric, i.e. $g'_{rs} = \tilde{g}'_{rs} = G_{rs}$. When branons are present, the induced metric is given by

$$g'_{rs} = \tilde{g}'_{rs} - \partial_r X^i \partial_s X^j \tilde{g}_{ij} \tag{39}$$

On the other hand, the action including the scalar curvature term is given by:

$$S_B = -f^4 \int_{\mathbf{R}\times B} dtd^3y\sqrt{g'} + \lambda f^2 \int_{\mathbf{R}\times B} dtd^3y\sqrt{g'}R', \tag{40}$$

where R' is the induced curvature on the wrapped brane and the volume term is now finite for fixed time. For small excitations, the effective action becomes

$$S_{eff}[X] = S^{(0)}_{eff}[X] + S^{(2)}_{eff}[X] + ... \tag{41}$$

Where the effective action for the branons up to $\mathcal{O}(p^2)$ is nothing but the non-linear sigma model corresponding to a symmetry-breaking pattern $G(M_3) \rightarrow H'$:

$$S^{(0)}_{eff}[X] \;=\; -f^4 \int_{\mathbf{R}\times B} dtd^3y\sqrt{\tilde{g}'}, \tag{42}$$

$$S^{(2)}_{eff}[X] \;=\; \frac{f^4}{2} \int_{\mathbf{R}\times B} dtd^3y\sqrt{\tilde{g}'}\tilde{g}_{ij}\tilde{g}'^{rs}\partial_r X^i \partial_s X^j + \lambda f^2 \int_{\mathbf{R}\times B} dtd^3y\sqrt{\tilde{g}'}\tilde{R}'. \tag{43}$$

Where $\tilde{R}'$ is the background curvature on the wrapped brane, without excitations. Notice that $i, j, ...$ are M_3 indices, whereas $r, s, ...$ are indices on the $\mathbf{R} \times K'$ manifold. This effective action is again an expansion in powers of $p \sim \partial_r X \sim \partial_r g'/f$, i.e. it is a low-energy expansion.

For static configurations the mass of the wrapped state, to the lowest order, is given from (40) again by

$$M_W = f^4 \int_B d^3y\sqrt{g'}. \tag{44}$$

The minimum is found for $X^i = 0$:

$$M_W = 2\pi^2 f^4 R_B^3, \tag{45}$$

which is proportional to the $B = S^3$ volume as expected. In this case we have the brane wrapped around B with the minimal possible brane volume. For small enough λ, adding the curvature term does not change the picture very much:

$$M_W = f^4 \int_B d^3y\sqrt{g'} - \lambda f^2 \int_B d^3y\sqrt{g'}R'. \tag{46}$$

Because of the scalar curvature on a 3-sphere is $\tilde{R}' = -6/R_B^2$, we find

$$M_W = 2\pi^2 f^4 R_B^3 \left(1 + 6\frac{\lambda}{R_B^2 f^2}\right). \tag{47}$$

Then the brane is still wrapped minimizing its volume but we have a new contribution to the mass coming from the brane curvature which coincides with the B curvature. This result obviously applies to branes wrapped once around B. The generalization to cases where the brane is wrapped $n_W \in \mathbf{Z}$ times is straightforward resulting just in a factor of $|n_W|$ in the above equation.

It is very interesting to realize that the obtained value for the wrapped-state mass is exactly the same previously given for the brane-skyrmion mass in Table.1, as the upper bound for positive λ and the exact value for negative λ, provided $\lambda > -R_B^2 f^2/6$. The fact that our brane action is defined in an entirely geometrical way, makes possible to give a beautiful explanation of this fact. On the right of Fig.2 we have represented a brane-skyrmion corresponding to a positive value of λ. According to our previous discussion the brane-skyrmion has a non-zero size. This makes possible to pass through the brane from one side to the other, showing the topological defect as some kind of hole in the brane.

Despite the similarity between both configurations, they are not the same because their topology is different. The brane-skyrmion is extended on both the compactified M_3 space and the extra-dimensional space B, but the wrapped states do only around the extra dimension space B. Another way to understand why they are different is to realize that the brane-skyrmion is made of a single piece unlike the wrapped configuration which has two different pieces (the wrapped brane and the world brane). Thus they cannot be connected by a classical process, although quantum tunneling could produce in principle transitions between one to the other.

8 Summary and conclusions

In this work we have studied the effective action describing the low-energy dynamics of the GB, which appear when the higher-dimensional space-time manifold isometry group is spontaneously broken by the presence of a three-brane Universe. From the $3+1$-dimensional point of view, those GB can be considered as some kind of new scalar fields whose dynamics is given by the non-linear sigma model lagrangian corresponding to the coset manifold $K = G/H$. Eventually, the GB can also get some mass terms due to small deviations from the simple ideal exact isometry pattern.

This spontaneous symmetry breaking gives rise, through the Higgs mechanism, to a mass matrix for the KK graviphotons associated to the isometries of the compactified space B. However, for the interesting case of large extra dimensions and $f \ll M_D$, the graviphotons decouple from the low-energy theory and their masses become very small. We can thus consider the GB as the only relevant degrees of freedom on the brane in the low-energy regime.

In order to make further studies of the possible phenomenological implications of those GB brane excitations, we have considered their corresponding couplings with the SM particles including scalars, fermions (chiral and non-chiral) and gauge bosons.

Under suitable assumptions about the third homotopy group of the space K, this effective action gives rise to a new kind of states corresponding to topological defects of the brane (brane-skyrmions) which are stable whenever the curvature parameter λ is not too negative. The mass and the size of the brane-skyrmions can be computed in terms of the brane tension scale (f), λ and the size of the space B (R_B). The brane-skyrmions can be understood as some kind of holes in the brane that make possible to pass through them along the B space. This is because in the core of the topological defect the symmetry is restablished. In the case considered here the broken symmetry is basically the translational symmetry along the extra-dimensions. Thus the core of the brane-skyrmion plays the role of a window through the brane, which is a nice geometrical interpretation of this object. For $\lambda = 0$ or negative the brane-skyrmion collapses to zero size and that window is closed.

Brane-skyrmions can in principle be quantized as bosons or fermions by adding a Wess-Zumino-Witten-like term to the branon effective action. This is a very interesting possibility since it provides a completely new way of introducing fermions on the brane. The low-energy effective lagrangian describing the interactions between branons and brane-skyrmions can also be obtained in a systematic way. This open the door for the study of the possible phenomenology of these states at the Large Hadron Collider (LHC) currently under construction at CERN.

We have studied another different set of states corresponding to a brane wrapped on the extra-dimension space B (wrapped states) and we have derived their connection with the brane-skyrmion states.

Finally, these results can be extended also to higher dimensions where similar results hold. This fact could have some relevance in the context of pure M-theory where solitonic 5-branes are present which could wrap around 5-dimensional spheres.

We understand that the brane-skyrmions and wrapped states studied in this paper are quite interesting objects (both from the theoretical and perhaps from a more phenomenological point of view) and thus we think that they deserve future research.

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